3401 2010

1 (a) $y'' - 2y' + y = f(x) \quad (*) \qquad y(0) = y'(0) = 0.$

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$y = Ae^x + Bxe^x \quad (+)$$

Try $y = A(x)e^x + B(x)e^x x$

$$y' = Ae^x + A'e^x + B'e^x x + Be^x x + Be^x$$

$$\cancel{y'' = Ae^x + 2A'e^x + A''e^x + B''e^x x + B'e^x x + B'e^x + B'e^x x + Be^x x + Be^x + B'e^x + Be^x}$$

Set $A'e^x + B'e^x x = 0 \Rightarrow A' + B'x = 0 \quad (1)$

$$y' = Ae^x + Be^x x + Be^x$$
$$y'' = A'e^x + Ae^x + B'e^x x + Be^x x + Be^x + B'e^x + Be^x$$

$(*)$ $A'e^x + \cancel{Ae^x} + B'e^x x + \cancel{Be^x x} + \cancel{Be^x} + B'e^x + \cancel{Be^x}$
$- \cancel{2Ae^x} - 2\cancel{Be^x}x - \cancel{2Be^x} + \cancel{Ae^x} + \cancel{Be^x}x = f(x)$

$$A' + B'x + B' = f(x)e^{-x}$$
$$A' + (1+x)B' = f(x)e^{-x} \quad (2)$$
$$A' + xB' = 0 \quad (1)$$

(2) − (1): $B' = f(x)e^{-x} \qquad \Rightarrow B = \int_0^x f(u)e^{-u}\,du$

(2) + (1) $\quad 2A' + B' + 2xB' = f(x)e^{-x}$

$$2A' + (1+2x)f(x)e^{-x} = f(x)e^{-x} \, (1)$$
$$2A' = -2xf(x)e^{-x}$$
$$A' = -xf(x)e^{-x} \quad \Rightarrow \quad A = -\int_0^x uf(u)e^{-u}\,du$$

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$(*)$: $y = Ae^x + Bxe^x$

$$= -\int u f(u) e^{x-u}\, du + \int f(u) x e^{-u+x}\, du$$

$$= \cancel{*}\int (x-u) e^{x-u} f(u)\, du$$

(b) (i) $\int_{C_i} e^{xt} f(t)\, \cancel{dt} = y$

$$x y'' + (n+1-x) y' - ny = 0$$

$$: \int (xt^2 + (n+1-x)t - n) e^{xt} f(t)\, dt = 0$$

If this is $\int_\gamma \underbrace{\frac{d}{dt}(e^{xt} g)}_{= xe^{xt} g + e^{xt} g'} dt$

then $f(t^2 - t) = g = f t(t-1)$

$f((n+1)t - n) = g'$

$$\frac{g'}{g} = \frac{(n+1)t - n}{t^2 - t} = \frac{A}{t} + \frac{\cancel{A}B}{t-1}$$

$$A(t-1) + Bt = (A+B)t - A$$

$$\Rightarrow A = n$$
$$B = 1$$

$$\frac{g'}{g} = \frac{n}{t} + \frac{1}{t-1}$$

$$\ln g = n \ln t + \ln(t-1) + a$$

$$g = Ct^n(t-1) \quad \rightarrow \quad f = Ct^{n-1}$$

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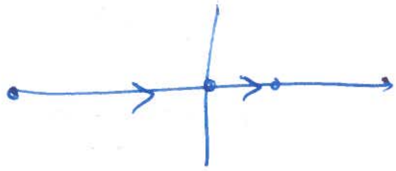
$$y = C\int_{C_i} e^{xt} t^{n-1}\, dt$$

Choose C_i s.t. $y \neq 0$ and $\left[e^{zt} g\right]_{C_i} = \left[e^{zt} t(t-1) f\right]_{C_i} = 0$

$t = 0, 1, -\infty$



$$y_1 = C\int_{-\infty}^{0} e^{xt} t^{n-1}\, dt \overset{s=-t}{=} -(-1)^{n-1} C\int_{\infty}^{0} e^{-xs} s^{n-1}\, ds$$

$$y_2 = C\int_{0}^{1} e^{xt} t^{n-1}\, dt \qquad " \; C\int_0^\infty e^{-xs} s^{n-1}\, ds \cdot (-1)^{n-1}$$

bounded as $x \to \infty$.

$$\Rightarrow y(x) = C\int_0^\infty e^{-xs} s^{n-1}\, ds. \quad (**)$$

$$y(1) = C\int_0^\infty e^{-s} s^{n-1}\, ds$$

$(**)$: Let $t = xs$

$$y(x) = C\int_0^\infty e^{-t} \frac{t^{n-1}}{x^{n-1}} \frac{dt}{x} \qquad \text{Singular at } x = 0$$

$$y(1) = C\int_0^\infty e^{-t} t^{n-1}\, dt.$$

$$\Rightarrow \underline{x^n y(x) = y(1)}.$$

2. (a) (i) phase-plane is xy plane. $\frac{dy}{dx} = \frac{Q}{P}$.

$\dot{x}$ and $\dot{y}$ are parametric equations of curves (trajectories) in the plane.

A periodic sol^n has $x(t_0+T) = x(t_0)$, $y(t_0+T) = y(t_0)$ some ~~some~~ T and all ~~to~~ t_0.

These will be closed trajectories.

(ii)
$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = \int_{\partial D} (P\,dy - Q\,dx) \quad \text{[Stokes]}$$
$$= \int_{\partial D} \left(P\frac{dy}{dt} - Q\frac{dx}{dt}\right) dt$$
$$= \int_{\partial D} (PQ - QP)\,dt = 0.$$

This is impossible if ∂D lies in a region where $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ does not change sign.

(b) (i) $\ddot{x} + \dot{x} + x - x^3 = 0$

If $\dot{x} = y$, $\ddot{x} = \frac{dy}{dt} = \dot{y}$

$\Rightarrow \dot{y} + y + x - x^3 = 0.$

$P = \frac{dx}{dt} = y$

$Q = \frac{dy}{dt} = -y - x + x^3$

(ii) No periodic sol^{n}s if $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ does not change sign.

$\frac{\partial P}{\partial x} = 0$

$\frac{\partial Q}{\partial y} = -1$

Sum $= -1$ does not change sign.

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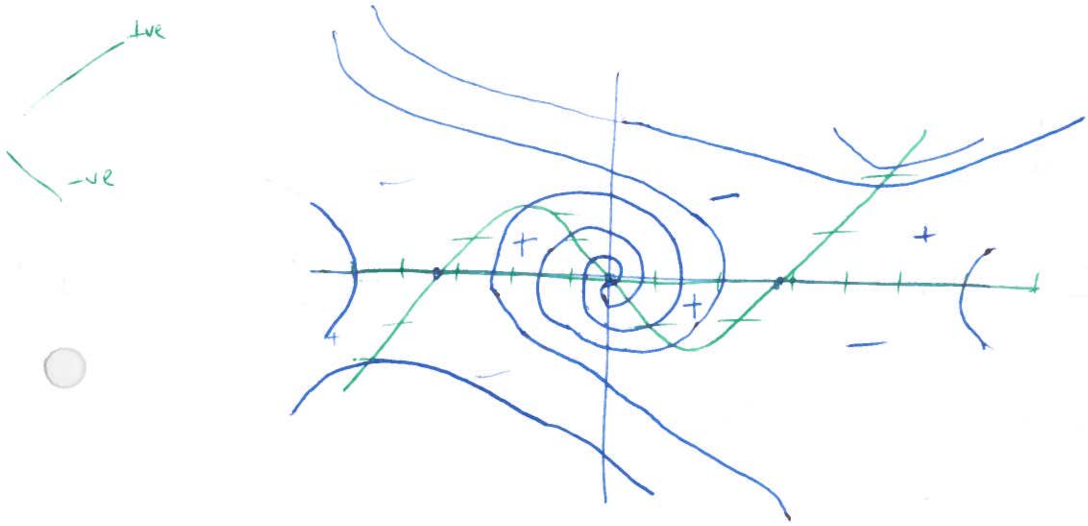
Nullclines: $P \overset{\frac{dx}{dt}}{=} 0 \quad \rightarrow \quad y = 0$

$Q = 0 \quad \rightarrow \quad -y - x + x^3 = 0$

$y = -x + x^3$

$y = x(-1 + x^2) \qquad x = 0, 1, -1$



Sing points where they meet $\quad (0,0)$, $(1,0)$, $(-1,0)$

$$J = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1+3x^2 & -1 \end{pmatrix}$$

$J|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. evals: $\begin{vmatrix} \lambda & -1 \\ 1 & \lambda+1 \end{vmatrix} \Rightarrow \lambda(\lambda+1) + 1 = 0$

$\cancel{(\lambda+1)^2 = 0} \rightarrow \cancel{\lambda = -1}$

$\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{1-4}}{2}$ $\quad$ ~~real~~, neg $\rightarrow$ stable ~~node~~ spiral

complex

$J|_{(1,0)} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$: evals

$\|$

$J|_{(-1,0)} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$

$\begin{vmatrix} \lambda & -1 \\ -2 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1) - 2 = 0$

$\lambda^2 + \lambda - 2 = 0$

$\lambda = \frac{-1 \pm \sqrt{1+8}}{2}$ $\quad$ real, diff. sign $\rightarrow$ saddle

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-1 \cancel{*} -1 + 1}{1} = \text{neg}.$

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3. $$\frac{d^2x}{dt^2} + \varepsilon(x^n - 1)\frac{dx}{dt} + x = 0 \qquad (*)$$

(a) $x(t) = a\cos\theta + \varepsilon x_1(\theta) + \ldots$

$\theta = (1 + \varepsilon n_1 + \cdots)t$

$$\frac{d}{dt} = \frac{d}{d\theta}\frac{d\theta}{dt}$$
$$= (1+\varepsilon n_1)\frac{d}{d\theta}$$

$$(*):\ (1+\varepsilon n_1)^2 \frac{d^2}{d\theta^2}(a\cos\theta + \varepsilon x_1) + \varepsilon(x^n-1)(1+\varepsilon n_1)\frac{d}{d\theta}(a\cos\theta + \varepsilon x_1)$$
$$+ a\cos\theta + \varepsilon x_1 = 0$$

$[\varepsilon^0]$ $-a\cos\theta + a\cos\theta = 0$ ✓

$[\varepsilon^1]$ $-2n_1 a\cos\theta + x_1'' - (x^n-1)a\sin\theta + x_1 = 0$

$$x_1'' + x_1 = 2n_1 a\cos\theta + (x^n - 1)a\sin\theta$$

no periodicity wanted on RHS — no sin/cos's. So set $n_1 = 0$ to get rid of cos.

Then $\times \sin$: $\int_{-\pi}^{\pi} (x^n-1)a\sin^2\theta\, d\theta = 0$

$$\int_{-\pi}^{\pi} (a^n\cos^n\theta - 1)\sin^2\theta\, d\theta = 0$$

$$\int_{-\pi}^{\pi} a^n\cos^n\theta\sin^2\theta\, d\theta = \int_{-\pi}^{\pi}\sin^2\theta\, d\theta = \pi$$

$$a^n\int_{-\pi}^{\pi}\cos^n\theta\sin^2\theta\, d\theta = \pi$$

$$\int_{-\pi}^{\pi} \cos^n\theta \sin^2\theta \, d\theta = \int_{-\pi}^{\pi} \cos^n\theta (1-\cos^2\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \cos^n\theta \, d\theta - \int_{-\pi}^{\pi} \cos^{n+2}\theta \, d\theta$$

If $n =$ odd, we get $0 = \pi$ ✱.

If $n =$ even this $= \dfrac{2\pi n!}{2^n[(\frac{n}{2})!]^2} - \dfrac{2\pi(n+2)!}{2^{n+2}[(\frac{n+2}{2})!]^2}$

$$= \quad \ldots \quad - \frac{2\pi n! \; (n+1)(n+2)}{2^n [(\frac{n}{2}+1)!]^2 \; 2^2}$$

$$- \frac{2\pi n! \; (n+1)(n+2)}{2^n [(\frac{n}{2})]^2 \; 2^2 \; (\frac{n}{2}+1)^2}$$

$$= \frac{2\pi n!}{2^n[(\frac{n}{2})!]^2}\left(1 - \frac{(n+1)(n+2)}{4(\frac{n}{2}+1)^2}\right)$$

$$\Rightarrow a = \cancel{2\pi}\left(\frac{\pi}{\text{that}}\right)^{1/n}$$

$$a^n = \frac{2^n[(\frac{n}{2})!]^2}{2n!}\left[\frac{4(\frac{n}{2}+1)^2 - (n+1)(n+2)}{4(\frac{n}{2}+1)^2}\right]^{-1}$$

$$= \frac{2^{n-1}[(\frac{n}{2})!]^2}{n!}\left[\frac{(n+2)^{\cancel{2}} - \cancel{(n+2)}(n+1)}{(n+2)^{\cancel{2}}}\right]^{-1}$$

$$\Rightarrow a = \left(\frac{2^{n-1}(n+2)[(\frac{n}{2})!]^2}{n!}\right)^{1/n}.$$

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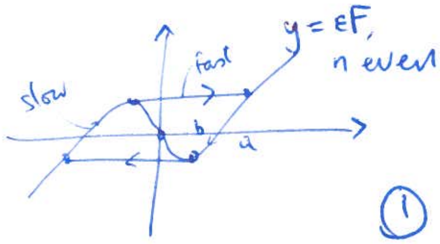
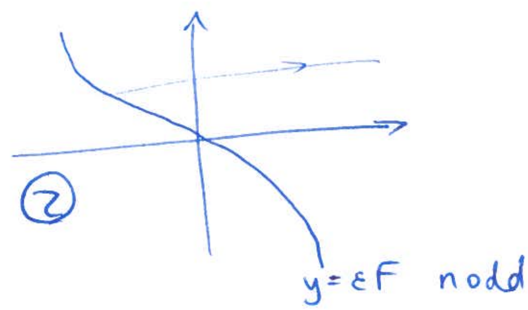
(b) $y(t) = \dfrac{dx}{dt} + \varepsilon F \qquad F = \dfrac{x^{n+1}}{n+1} - x$

$\Rightarrow \dfrac{dy}{dt} = \dfrac{d^2x}{dt^2} + \varepsilon F' \dfrac{dx}{dt} \qquad F' = x^n - 1 .$

$\Rightarrow$ can write

$$\frac{dx}{dt} = y - \varepsilon F(x) \qquad \frac{dy}{dt} = -x$$





If $\varepsilon \gg 1$ and y is not close to εF,
then $\dfrac{dx}{dt} = O(\varepsilon) \gg 1$.

So x increases rapidly, but ~~as~~ as x must remain finite, it can't do so for long.

So as x increases rapidly, y remains approximately constant since $dy/dt = -x$, which is finite.

So trajectories are approximately $y = \text{const}$.

However, $\dot{x}$ & $\dot{y}$ are not large if $y = \varepsilon F$. So closed trajectories are possible as in ①, but not in ②.

Period is $2\int_a^b dt = 2\int \dfrac{dt}{dx}\,dx = 2\int \dfrac{dt}{dy}\dfrac{dy}{dx}\,dx$

$= 2\int_a^b \dfrac{F'(x)}{-x}\,dx = 2\varepsilon \int_b^a \dfrac{x^n - 1}{x}\,dx$ etc.

4. $\ddot{x} + x = \varepsilon f(\dot{x})$ $(*)$

$x = x_0 + \varepsilon x_1$

$T = \varepsilon t$

$\hookrightarrow \frac{d}{dt} \to \frac{\partial}{\partial t} + \frac{\partial}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}$

$(*)$: $\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial T} + \varepsilon^2 \frac{\partial^2}{\partial T^2}\right)(x_0 + \varepsilon x_1)$

$+ \cancel{\left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}\right)}(x_0 + \varepsilon x_1) = \varepsilon f(\dot{x})$

$[\varepsilon^0]$ $\frac{\partial^2}{\partial t^2} x_0 + x_0 = 0 \Rightarrow x_0 = A(T)\sin(t + \Phi(T))$

$[\varepsilon^1]$ $\frac{\partial^2}{\partial t^2} x_1 + 2\frac{\partial^2}{\partial t \partial T} x_0 + x_1 = f(\dot{x})$

$\frac{\partial}{\partial t} x_0 = A(T)\cos(\overbrace{t + \Phi(T)}^{\chi})$

$\frac{\partial^2}{\partial t \partial T} x_0 = A'\cos\chi - A\sin\chi\,\Phi'$

$\frac{\partial^2 x_1}{\partial t^2} + x_1 = f(\dot{x}) - 2A'\cos\chi + 2A\sin\chi\,\Phi'$

$\times\cos\int$: $0 = \int \cos\chi\, f(\dot{x})\, d\chi - 2A'\pi$

$\Rightarrow A' = \frac{1}{2\pi}\int \cos\chi\, f(A\cos\chi)\, d\chi$

$\times\sin\int$ $0 = \int \sin\chi\, f(\dot{x})\, d\chi + 2A\Phi'\pi$

$\Rightarrow A\Phi' = -\frac{1}{2\pi}\int \sin\chi\, f(A\cos\chi)\, d\chi$

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(b) $\frac{d\Phi}{dT}$ is integral over a period of an odd × even

$= 0.$

(c) $f(u) = u - \alpha u^n$

$$\frac{dA}{dT} = \frac{1}{2\pi}\int_0^{2\pi} \cos x\left[A\cos x - \alpha A^n \cos^n x\right] dx$$

$$= \frac{1}{2\pi}\left[A\pi - \alpha A^n \int \cos^{n+1} x \, dx\right]$$

$$= \frac{A}{2} - \frac{\alpha A^n}{2\pi}(0)$$

n even $\Rightarrow n+1$ odd

$$\Rightarrow \frac{dA}{dT} = \frac{A}{2}$$

$$\Rightarrow \int \frac{1}{A} dA = \int \frac{1}{2} dT$$

$$A = Ce^{T/2}$$

$\frac{dA}{dT} \neq 0$ unless $x = 0$. $\Rightarrow$ no limit cycle

n odd.

$$\frac{dA}{dT} = \frac{A}{2} - \frac{\alpha A^n}{2\pi} \frac{2\pi (n+1)!}{2^{n+1}\left(\frac{n+1}{2}!\right)^2}$$

$$= \frac{A}{2} - \frac{\alpha A^n (n+1)!}{2^{n+1}\left(\frac{n+1}{2}!\right)^2}$$

$$= \frac{A}{2}\Bigg(1 - \underbrace{\frac{A^{n-1}\alpha (n+1)!}{2^n \left(\frac{n+1}{2}!\right)^2}}_{+ve}\Bigg)$$

$$\Rightarrow \text{when } \left(\frac{2^n\left(\frac{n+1}{2}\right)!^2}{\alpha (n+1)!}\right)^{\frac{1}{n-1}} = A. \quad \text{"}A_p\text{"}$$

oh lordy

(for $n \neq 1$).

when $A > A_p$, $\frac{dA}{dT} < 0$

$A < A_p$ $\frac{dA}{dT} > 0.$

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$n=1, \quad \frac{dA}{dT} = \frac{A}{2}\left(1 - \frac{\alpha 2}{2}\right) = \frac{A}{2}(1-\alpha)$

$n=1$ doesn't work but that is to be expected.

and $x(t) = A_p \sin(t+\phi) \quad \because \quad \Phi$ indpt of T.

not $\Phi(T)$

---

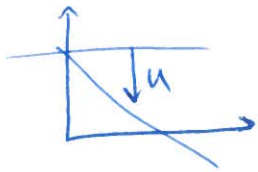
5. (a) (i) If $I(x) = \int_0^T e^{-xt} f(t)\, dt$

and $f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^n$ as $t \to 0$

and $f(t)$ does not grow superexponentially if $T=\infty$

as $t \to 0$

$$\Rightarrow I(x) \sim \sum_{n=0}^\infty \frac{a_n(\alpha+n)!}{x^{n+\alpha-1}}$$



(ii) Let $u(t) = \phi(a) - \phi(t)$

Limits a: $u=0$

b: $u = \phi(a) - \phi(b) =: \beta$

dt: $u = \phi(a) - \phi(t)$

$du = -\phi'(t)\, dt$

$$\Rightarrow I(x) = \int_0^\beta e^{x\phi(a)} e^{-xu} f(t) \frac{1}{-\phi'(t)}\, dt$$

Near $t=a$, $u=0$ and $\frac{f(t)}{-\phi'(t)} = \frac{f(a)}{-\phi'(a)}$

$$\Rightarrow I(x) \sim \underline{e^{x\phi(a)}\left(-\frac{1}{x}\right)\frac{f(a)}{\phi'(a)}} \quad \text{by Watson's lemma.}$$

If max at b, result is $\frac{e^{x\phi(b)} f(b)}{x\,\phi'(b)}$.

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(iii) Focus on region near max. of ϕ and

$$\phi(t) = \phi(a) + \cancel{\phi'(a)}(t-a) + \frac{1}{2}\phi''(a)(t-a)^2$$

$$\Rightarrow \quad e^{x\phi(t)} \sim e^{x\phi(a)} e^{-\frac{1}{2}|\phi''(a)|\, x\,(t-a)^2}$$

Let $u^2 = \frac{1}{2}|\phi''(a)| \, x \, (t-a)^2$

$$\Rightarrow \cancel{2}u\, du = \frac{1}{2}|\phi''(a)| \, x \, \cancel{2}(t-a)\, dt$$

$$- \quad dt = \frac{2u \quad du}{|\phi''(a)| x (t-a)}$$

$$= \frac{\cancel{2}\left[\frac{1}{2}|\phi''(a)| \, x\, (t-a)^2\right]^{1/2}}{\frac{1}{2}|\phi''(a)| \, x\,(t-a)} du$$

$$= \sqrt{\frac{2}{x|\phi''(a)|}} \; \frac{t-a}{t} \quad du$$

$$\Rightarrow \quad I(x) = \int_0^\infty e^{x\phi(a)} e^{-u^2} f(a) \sqrt{\frac{2}{x|\phi''(a)|}} \, du$$

$$= \frac{1}{2}\left[e^{x\phi(a)} \sqrt{\frac{2}{x|\phi''(a)|}} \int_{-\infty}^{\infty} e^{-u^2} du \quad f(a) \right]$$

$$= e^{x\phi(a)} \sqrt{\frac{\cancel{2}\pi}{2|\phi''(a)|}} \; f(a) .$$

(b) (i) $\int_0^\infty e^{-xt^2} \sin t \, dt$

$u = t^2$ $\quad$ $t = u$

$du = 2t \, dt$

$\frac{1}{2u^{1/2}} du = dt$

$$= \int_0^\infty e^{-xu} \underbrace{\frac{\sin(u^{1/2})}{2u^{1/2}}}_{f} du$$

$f \sim \frac{1}{2} u^{-1/2} \left(u^{1/2} + \cdots \right)$ $\qquad \lambda = -\frac{1}{2}$ $\qquad a_0 =$

$\rightarrow \quad I(x) \sim \frac{1}{2x}.$

(ii) $\int_{-\pi/2}^{\pi/2} (t+2) e^{-x\cos t} \, dt \quad = \int_{-\pi/2}^{0} (t+2) e^{-x\cos t} \, dt + \underbrace{\int_{0}^{\pi/2} (t+2) e^{-x\cos t} \, dt}$

$= \int_{-\pi/2}^{0} (2-t) e^{-x\cos t} \, dt$

$\phi(t) = -\cos t$

$f(t) = t+2$

ϕ not decreasing here $\rightarrow$

$\Rightarrow \quad a = -\pi/2$

$\cos(-\pi/2) = 0$

$\phi'(a) = -1$

$\phi(t) = -\cos t$

$f(t) = 2 - t$

$\rightarrow$

$$\sim -\left(\frac{-\frac{\pi}{2} + 2}{-x} \right) - \left(\frac{+\frac{\pi}{2} + 2}{-x} \right)$$

$$= \underline{\underline{\frac{4}{x}}}.$$

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(iii) $\int_0^\infty e^{(x^2t^2-t^4)} g\left(\frac{t}{x}\right) dt$

$u = \frac{t}{x}$

$du = \frac{1}{x} dt$

$\int_0^\infty e^{x^4(u^2-u^4)} g(u)\, x\, du$

$t^4 = x^4 u^4$

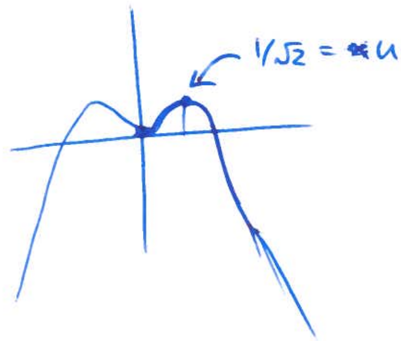
$x^2 t^2 = x^4 u^2$

Let $y = x^4$

$\int_0^\infty e^{y(u^2-u^4)} g(u)\, y^{1/4}\, du$

$= y^{1/4} \int_0^\infty e^{y(u^2-u^4)} g(u)\, du$

$u^2(1-u^2)$



$\phi(u) = u^2 - u^4$ $\quad f(u) = g(u)$

$\phi'(0) = 0$

$\phi''(0) = 2 \neq 0$

$a = \frac{1}{\sqrt{2}}$ $\quad \phi'(a) = 0$ $\quad \phi'' = 2 - \frac{12}{2} = -4$

$\phi(a) =$

$\sim y^{1/4}\, e^{1/4\, y}\, g\left(\frac{1}{\sqrt{2}}\right) \sqrt{\frac{2\pi}{4x}}$

$= \sqrt{\frac{\pi}{2}}\, \frac{e^{\frac{1}{4}x^4}}{x}\, g\left(\frac{1}{\sqrt{2}}\right)$

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